Q1 The "folding" and the "tunneling" techniques

Sunday, November 22, 2015 11:14 AM

In parts (a) and (b), we consider $\cos(2\pi f_0 t)$.

To find the perceived freq., we will use the "folding technique":

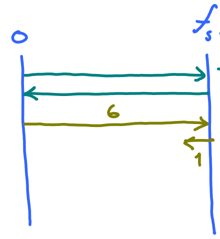
i) Consider the window of freq. from 0 to $\frac{f_s}{2}$.

ii) Start from 0, increase the freq. to f_0

Fold back at 0 and $\frac{f_s}{2}$ if necessary.

(a) $f_0 = 1{,}111{,}111$

$f_s/2 = \frac{12}{2} = 6$



} repeat this $\left\lfloor \frac{f_0}{f_s} \right\rfloor$ times $= 92592$

Alternatively,

$$12 \overline{)1{,}111{,}111} = 92592$$

108
31
24
71
60
111
108
31
24
7

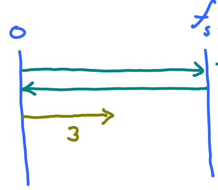
Remainder $= f_0 - f_s \left\lfloor \frac{f_0}{f_s} \right\rfloor$

$= 7 \leftarrow$ still $> \frac{f_s}{2}$

$f_r = \frac{f_s}{2} - 1 = 6 - 1 = 5$ Hz

(b) $f_0 = 111{,}111$

$f_s/2 = \frac{12}{2} = 6$



} repeat this $\left\lfloor \frac{f_0}{f_s} \right\rfloor$ times $= 9259$

Alternatively,

$$12 \overline{)111{,}111} = 9259$$

108
31
24
71
60
111
108
3

Remainder $= f_0 - f_s \left\lfloor \frac{f_0}{f_s} \right\rfloor$

$= 3$

$f_r = 3$ Hz

In parts (c) and (d), we consider $e^{j2\pi f_0 t}$.

To find the "perceived" frequency, we will use the "tunneling technique":

i) consider the window of freq. from $-\frac{f_s}{2}$ to $-\frac{f_s}{2}$.

ii) Start from 0.

If $f_0 > 0$, increase the freq. to f_0 (going to the right)

restart at $-\frac{f_s}{2}$ when $\frac{f_s}{2}$ is reached.

This is the "tunneling" part.

If $f_0 < 0$, decrease the freq. to f_0 (going to the left)

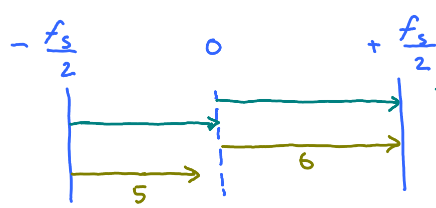
restart at $+\frac{f_s}{2}$ when $-\frac{f_s}{2}$ is reached.

(c) $f_0 = 11{,}111$

Remainder $= f_0 - f_s \left\lfloor \frac{f_0}{f_s} \right\rfloor$

$= 11$

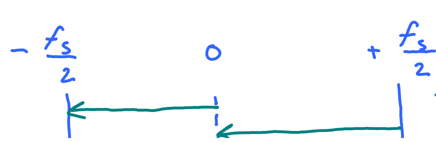
$f_r = -1$ Hz



} repeat this $\left\lfloor \frac{f_0}{f_s} \right\rfloor$ times $= 925$

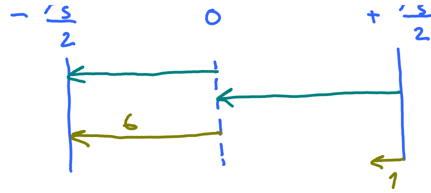
(d) $f_0 = -1{,}111$

Remainder $= 7$



} repeat this $\left\lfloor \frac{|f_0|}{f_s} \right\rfloor$ times

Remainder = 7



$\Big\}$ repeat this $\underbrace{\left\lfloor \frac{|f_0|}{f_s} \right\rfloor}_{=92}$ times

f_r = 5 Hz

Derivation of the "folding" and the "tunneling" techniques

Saturday, November 21, 2015 10:16 PM

Finding the "perceived" frequency of $g(t) = e^{j2\pi f_0 t}$ when the sampling rate is f_s:

Method 1: Analysis via the reconstruction equation.

First, note that $g(t) = e^{j2\pi f_0 t} \xrightarrow{\mathcal{F}} G(f) = \delta(f - f_0)$

With the sampling rate $= f_s$, we know that

$$G_\delta(f) = \sum_k f_s G(f - kf_s) = \sum_k f_s \delta(f - kf_s - f_0) = f_s \sum_k \delta(f - (f_0 + kf_s))$$

Recall that this is periodic with period f_s.

Now, the reconstruction equation gives

$$G_r(f) = LPF\{G_\delta(f)\} \text{ where } H_{LP}(f) = \begin{cases} T_s, & -\frac{f_s}{2} < f \le \frac{f_s}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

(or, equivalently, $g_r(t) = LPF\{g_\delta(t)\}$)

Therefore, only the parts of $G_\delta(f)$ that are between $\overbrace{-\frac{f_s}{2} < f \le \frac{f_s}{2}}^{\text{one period}}$ will survive the LPF (and will also be further scaled by T_s).

Our task now is then to find all value(s) of integer k such that

$$-\frac{f_s}{2} < f_0 + kf_s \le \frac{f_s}{2}$$

$$-\frac{1}{2} - \frac{f_0}{f_s} < k \le \frac{1}{2} - \frac{f_0}{f_s}$$

Note that the difference between these two numbers is one. So, there is exactly one value of k that satisfies such condition.

So, $k = \left\lfloor \frac{1}{2} - \frac{f_0}{f_s} \right\rfloor$

Note that "$\lfloor\ \rfloor$" is the "floor" function.

One may also consider $k = \left\lceil -\frac{1}{2} - \frac{f_0}{f_s} \right\rceil$. However, because the equality in the condition has equality at the upper-bound, the ceiling function of the lower bound will give the wrong answer when the lower bound is an integer itself.

This gives $G_r(f) = \underbrace{T_s}_{\text{gain the passband of the LPF}} \underbrace{f_s \delta(f - (f_0 + kf_s))}_{\text{the only term in } G_\delta(f) \text{ that is in the passband of the LPF}}$ where $k = \left\lfloor \frac{1}{2} - \frac{f_0}{f_s} \right\rfloor$

$$= \delta(f - (f_0 + kf_s))$$

$$\Big\downarrow \mathcal{F}^{-1}$$

$$g_r(t) = e^{j2\pi(f_0 + kf_s)} \quad \text{where } k = \left\lfloor \frac{1}{2} - \frac{f_0}{f_s} \right\rfloor.$$

Hence, the "perceived" frequency is $f_r = f_0 + kf_s = f_0 + f_s\left\lfloor \frac{1}{2} - \frac{f_0}{f_s} \right\rfloor$.

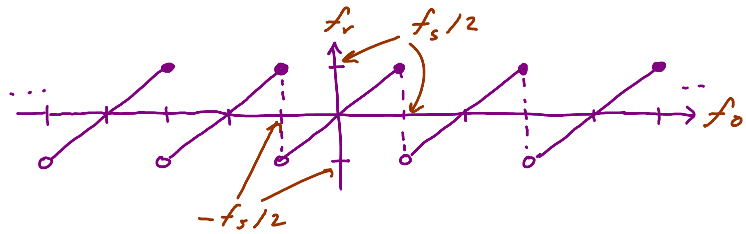
In part (c), $f_0 = 11{,}111 \Rightarrow k = \left\lfloor \frac{1}{2} - \frac{11{,}111}{12} \right\rfloor = \lfloor -925.41 \rfloor = -926$

$\Rightarrow f_r = 11{,}111 + (-926) \times 12 = -1$ Hz.

In part (d), $f_o = -1{,}111 \Rightarrow k = \left\lfloor \frac{1}{2} + \frac{1{,}111}{12} \right\rfloor = \lfloor 93.1 \rfloor = 93$

$\Rightarrow f_r = -1{,}111 + 93 \times 12 = 5$ Hz.

Remark: The plot of $f_r = f_o + f_s \left\lfloor \frac{1}{2} - \frac{f_o}{f_s} \right\rfloor$ is shown below:



Observe the "tunneling effect":

i) f_r is contained between $-\frac{f_s}{2}$ and $\frac{f_s}{2}$.

ii) $f_r = f_o$ in the above window

iii) when f_o exceed $\frac{f_s}{2}$, it "jumps" (or "goes through the tunnel symbolized by the dotted line") back to restart at $-\frac{f_s}{2}$.

iv) as a function of f_o, f_r is periodic with period f_s.

$\Rightarrow$ Therefore, instead of considering f_o, we may simply consider $f_t = \underbrace{f_o \bmod f_s}$

This is implemented in MATLAB by mod(f_o, f_s)

Alternatively, one can use

$$f_t = f_o - f_s \times \left\lfloor \frac{f_o}{f_s} \right\rfloor.$$

This leads to method 2 below

Method 2: Use the "tunneling effect" discussed in class (wherein we observe the location of the impulse(s) shown by our plotspect function).

I) Find $f_t = f_o \bmod f_s$ ← This gives $f_t \in [0, f_s)$

or, equivalently, $f_t = f_o - f_s \underbrace{\left\lfloor \frac{f_o}{f_s} \right\rfloor}$

Think of this as representing the number of "rounds" you start from 0, go through the tunnel, and back to 0.

II) $$f_r = \begin{cases} f_t, & \text{if } f_t \le \frac{f_s}{2}, \\ f_t - f_s, & \text{if } f_t > \frac{f_s}{2}. \end{cases}$$

In part (c), $f_o = 11{,}111 \Rightarrow f_t = f_o \bmod f_s = 11 > \overset{f_s/2}{6}$

$\Rightarrow f_r = f_t - f_s = 11 - 12 = -1.$

In part (c), $f_0 = 11{,}111 \Rightarrow f_t = f_0 \bmod f_s = 11 > 6$
$\Rightarrow f_r = f_t - f_s = 11 - 12 = -1.$

In part (d), $f_0 = -1{,}111 \Rightarrow f_t = f_0 \bmod f_s = 5 \leq 6$
$\Rightarrow f_r = f_t = 5.$

Finding the "perceived" frequency of $g(t) = \cos(2\pi f_c t)$ when the sampling rate is f_s.

Here, we write f_c instead of f_0 because there are two terms with two freq.: $f_0 = f_c$ and $f_0 = -f_c$

Method 1: From the Euler's formula: $g(t) = \cos(2\pi f_c t) = \frac{1}{2}e^{j2\pi f_c t} + \frac{1}{2}e^{j2\pi(-f_c)t}$.

After sampling, we can apply what we know about the "perceived" frequency of complex-exponential signal to get the "perceived" freq. of each term inside $g(t)$.

In part (a), $f_c = 1{,}111{,}111 \Rightarrow f_t = f_c \bmod f_s = 7 > 6$ ($f_s/2$)
$\Rightarrow f_r = f_t - f_s = 7 - 12 = -5.$

$-f_c = -1{,}111{,}111 \Rightarrow f_t = (-f_c) \bmod f_s = 5 \leq 6$
$\Rightarrow f_r = f_t = 5$

Therefore, $g_r(t) = \frac{1}{2}e^{j(2\pi(-5)t)} + \frac{1}{2}e^{j(2\pi(5))t} = \cos(2\pi(5)t).$

The "perceived" frequency is 5 Hz.

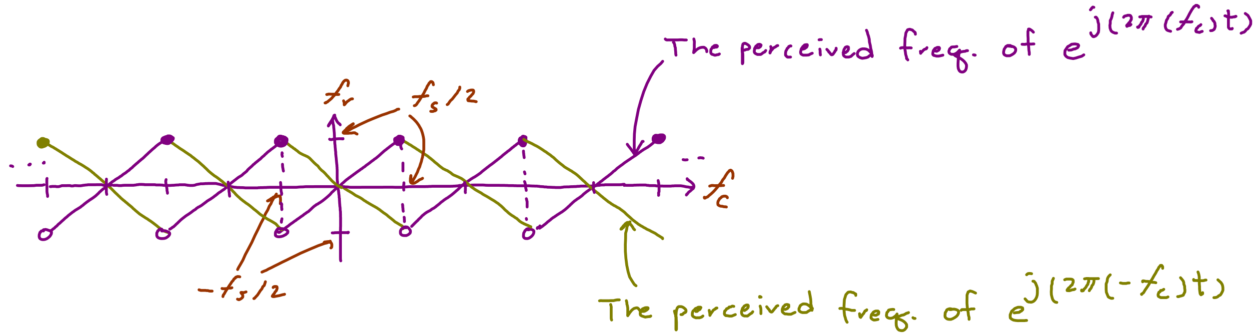
In part (b), $f_c = 111{,}111 \Rightarrow f_t = 3 \leq 6 \Rightarrow f_r = 3$

$-f_c = -111{,}111 \Rightarrow f_t = 9 > 6 \Rightarrow f_r = 9 - 12 = -3$

Therefore, $g_r(t) = \frac{1}{2}e^{j(2\pi(3)t)} + \frac{1}{2}e^{j(2\pi(-3)t)} = \cos(2\pi(3)t)$

The "perceived" freq. is 3 Hz.

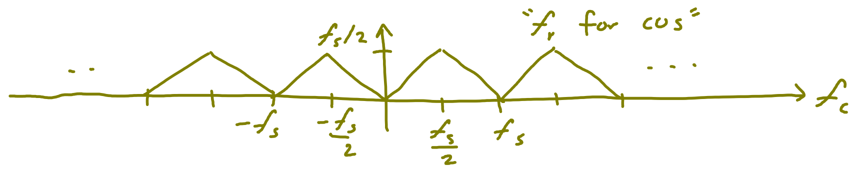
Method 2: When we consider the "perceived freq." of $e^{j2\pi f_c t}$ which we plotted earlier with " " $e^{j2\pi(-f_c)t}$



Observe that at every f_c,
for $\cos(2\pi f_c t)$
its two complex-expo components always gives a pair of

perceived freq., one positive and one negative
(except when f_c is a multiple of $f_s/2$)

so, the reconstructed signal $g_r(t)$ will still be a cosine whose freq. can simply be "read" from the upper part of the plot above:



(Because $\cos(-x) = \cos(x)$, we only answer one freq. for the cosine.)

Observe the "folding effect":

i) "f_r for cos" is contained between 0 and $\frac{f_s}{2}$

ii) " $= f_c$ in the above window

iii) when f_c exceeds $\frac{f_s}{2}$, it folds back towards 0.
when f_c reaches 0, it folds back towards $\frac{f_s}{2}$.

iv) as a function of f_c,
"f_r for cos" is periodic with period f_s.

Therefore, we can find "f_r for cos" by

I) find $f_t = f_c \bmod f_s = f_c - f_s \left\lfloor \frac{f_c}{f_s} \right\rfloor$.

II) $f_r = \begin{cases} f_t, & \text{if } f_t < \frac{f_s}{2}, \\ f_s - f_t, & \text{if } f_t > \frac{f_s}{2}. \end{cases}$

In part (a), $f_c = 1{,}111{,}111 \Rightarrow f_t = 7 > 6 \Rightarrow f_r = 12 - 7 = 5$ Hz.

In part (b), $f_c = 111{,}111 \Rightarrow f_t = 3 < 6 \Rightarrow f_r = 3$ Hz.

Problem 2: Aliasing and periodic square wave

First, let's recall some theoretical results we studied earlier. We know, from Example 4.20 in lecture, that

$$1[\cos\omega_0 t \geq 0] = \frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \frac{1}{7}\cos 7\omega_0 t + \cdots\right),$$

where $\omega_0 = 2\pi f_0$.

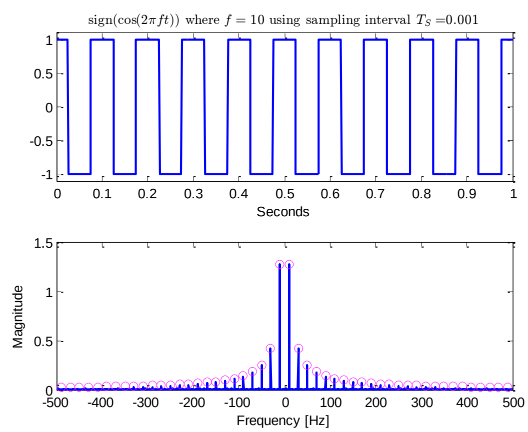
Here, we have a bipolar square pulse periodic signal $\mathrm{sgn}(\cos\omega_0 t)$ which alternates between "-1 and 1" instead of "0 and 1". Observe that

$$\mathrm{sgn}(\cos\omega_0 t) = 2 \times 1[\cos\omega_0 t \geq 0] - 1.$$

Therefore,

$$\mathrm{sgn}(\cos\omega_0 t) = \frac{4}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \frac{1}{7}\cos 7\omega_0 t + \cdots\right).$$

Hence, theoretically, its Fourier transform should have spikes (impulses) at all the odd-integer multiples of $\pm f_0$ Hz. The center spikes (at $\pm f_0$) should be the largest among them as shown in the Figure below.

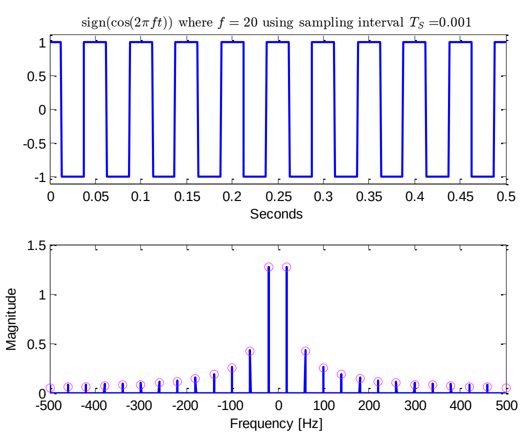


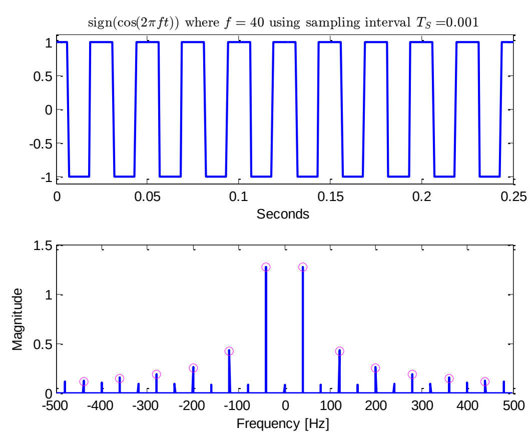
In fact, we could even try to predict the height of the spikes shown in the plot as well. Note that, in the script, we consider the time between 0 to 2 sec. Therefore, actually, we are not looking at the signal $\mathrm{sgn}(\cos\omega_0 t)$ from $-\infty$ to ∞. This time-limited view means that, in the frequency domain, we won't see the impulses but rather sinc pulses at those mentioned locations. (This is the same as seeing two sinc pulses instead of two impulses when looking at the Fourier transform of the cosine pulse.)

The sinc function is simply the Fourier transform of the rectangular windows. Because the area of the rectangular window is 1×2 = 2, its Fourier transform (which is a sinc function) has its peak value of 2. This is further scaled by a factor of ½ from the cosine. Therefore, each "impulse" ("sinc") that we see should have its height being the coefficient of corresponding cosine. For example, at $\pm f_0$, the coefficient of the cosine is $\frac{4}{\pi}$. Therefore, we expect the height of the "impulse" at $\pm f_0$ to be $\frac{4}{\pi} \approx 1.2732$. The height values for other impulse locations is shown by the pink circles in the plot. We see that our predicted values match the plot quite well.

In the time domain, the switching between the values -1 and 1 should be faster as we increase f_0. All the plots here are adjusted so that they show 10 periods of the "original signal" in the time domain. (This is done so that the distorted shape (if any) of the waveform in the time domain is visible.)

From the plots, as we increase f_0 from 10 to **20 Hz**, the locations of spikes changes from all the odd-integer multiples of 10 Hz to all the odd-integer multiples of 20 Hz. In particular, we see the spikes at ±20, ±60, ±100, ±140, ±180, ±220, ±260, ±300, ±340, ±380, ±420, ±460. Note that `plotspect` (by the way that it is coded) only plots from [$-f_s/2, f_s/2$). So, we see a spike at -500 but not 500. Of course, the Fourier transform of the sampled waveform is periodic and hence when we replicate the spectrum every f_s, we will have a spike at 500. Note that, in theory, we should also see spikes at ±540, ±580, ±620, ±660, and so on. However, because the sampling rate is 1000 [Sa/s], these high frequency spikes will suffer from aliasing and "fold back"[1] into our viewing window [$-f_s/2, f_s/2$). However, they fall back to the frequencies that already have spikes (for example, ±540 will fold back to ±460, and ±580 will fold back to ±420) and therefore the aliasing effect is not easily noticeable in the frequency domain.

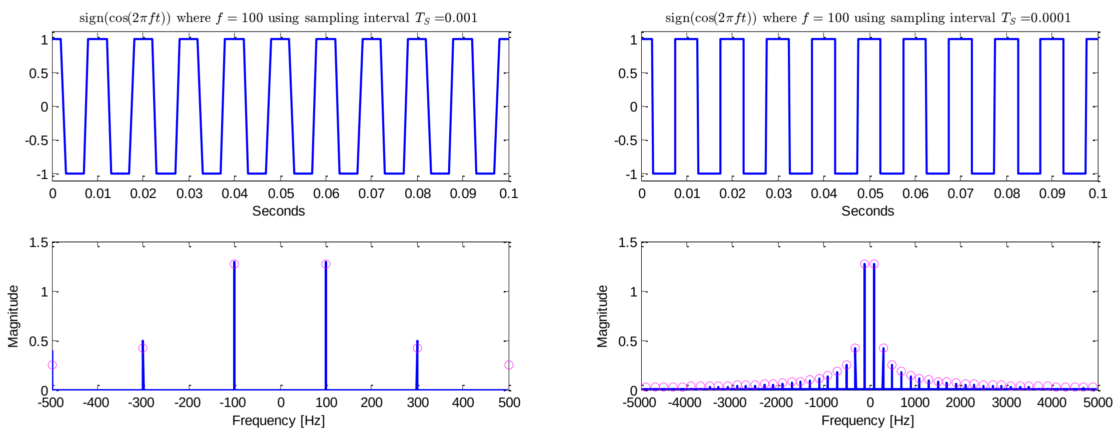




[1] Because the squarewave is real and even, the Fourier transform is also real and even. Therefore, the "folding effect" is equivalent to the "tunneling effect".

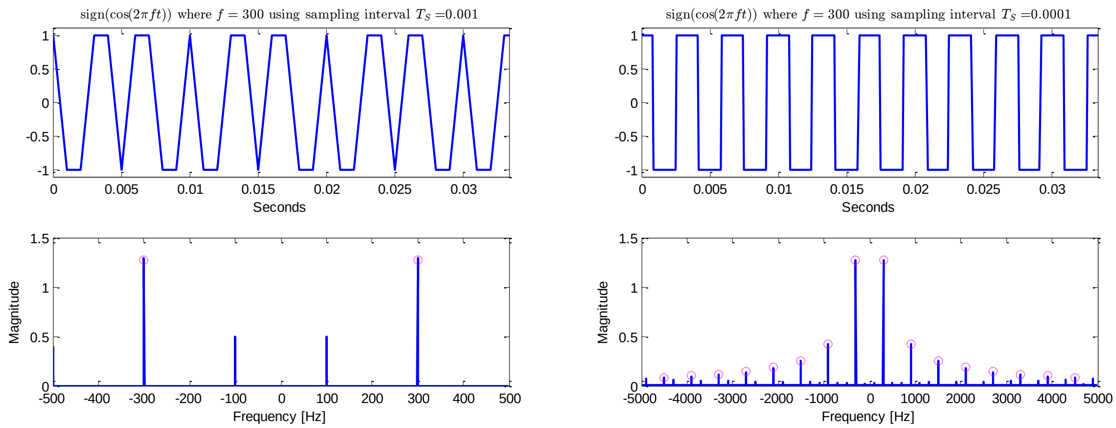
When f_0 = **40 Hz**, we start to see the aliasing effect in the frequency domain. Instead of seeing spikes only at ±40, ±120, ±200, ±280, ±360, ±440, the spikes at higher frequencies (such as ±520, ±600, and so on) fold back to lower frequencies (such as ±480, ±400, and so on). The plot in the time domain still looks quite OK with small visible distortion.

At high fundamental frequency f_0 = **100 Hz**, we see stronger effect of aliasing. In the time domain, the waveform does not look quite "rectangular". In the frequency domain, we only see the spikes at ±100, ±300, and 500. These are at the correct locations. However, there are too few of them to reconstruct a square waveform. The rest of the spikes are beyond our viewing window. We can't see them directly because they fold back to the frequencies that are already occupied by the lower frequencies. Note also that the predicted height (pink circles) at ±300 Hz is quite different from the `plotspect` value. This is because the content from the folded-back higher-frequencies is being combined into the spikes.



Our problem can be mitigated by reducing the sampling interval to T_S = 1/1e4 instead of T_S = 1/1e3 as shown by the plot on the right above.

Finally, at the highest frequency ***f_0* = 300 Hz**, if we still use T = 1/1e3, the waveform will be heavily distorted in the time domain. This is shown in the left plot below. We have large spikes at ±300 as expected. However, the next pair which should occur at ±900 is out of the viewing window and therefore folds back to ±100. Again, the aliasing effect can be mitigated by reducing the sampling time to T = 1/1e4 instead of T = 1/1e3. Now, more spikes show up at their expected places. Note that we can still see a lot of small spikes scattered across the frequency domain. These are again the spikes from higher frequency which fold back to our viewing window.



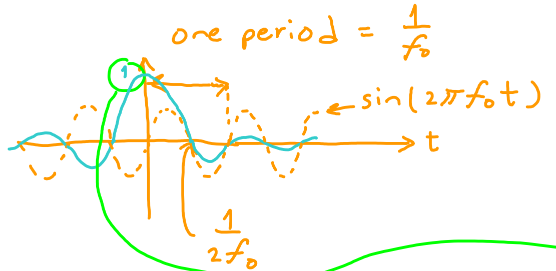
sign(cos(2πft)) where f = 300 using sampling interval T_S =0.001
sign(cos(2πft)) where f = 300 using sampling interval T_S =0.0001
Seconds
Seconds
Magnitude
Magnitude
Frequency [Hz]
Frequency [Hz]

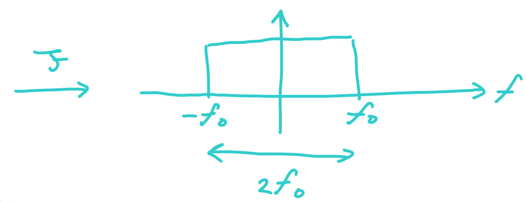
Q3 Nyquist sampling rate and Nyquist sampling interval

Sunday, July 17, 2011 2:09 PM

To apply the sampling theorem (which gives the Nyquist sampling rate and interval), we first need to find the value B where the signal in each part is bandlimited to.

The signals involved in this question are of the form $\text{sinc}(2\pi f_0 t)$.
Therefore, we first find a general result for $\text{sinc}(2\pi f_0 t)$.
First, we draw $\sin(2\pi f_0 t)$:





Then, we draw the $\text{sinc}(2\pi f_0 t)$ using the zeroes of $\sin(2\pi f_0 t)$
We then see that the first zero occurs at $\frac{1}{2f_0}$. Therefore, in the freq. domain, the corresponding rectangular function has width $= 2f_0$. So, its boundaries are $\pm f_0$.
Conclusion: $\text{sinc}(2\pi f_0 t)$ is bandlimited to $B = f_0$
Note that the height of the rectangular function must be $\frac{1}{2f_0}$ to make its area $= 1$.

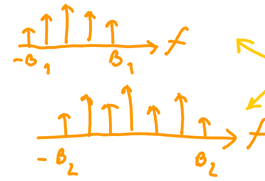
(a) $\text{sinc}(100\pi t) = \text{sinc}(2\pi \times 50 \times t) \Rightarrow B = 50\text{ Hz}$

(b) Recall that for signals $g_1(t)$ bandlimited to B_1 and
$g_2(t)$ " " B_2,

their product $g_1(t)g_2(t)$ is bandlimited to $B_1 + B_2$.
To "see" this (without actually doing the "flip-shift-integrate" for convolution in the freq. domain, imagine

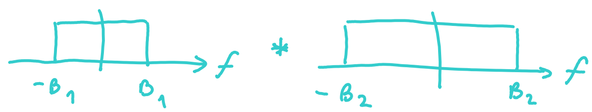
$G_1(f)$ as a bunch of impulses

$G_2(f)$ as a bunch of impulses

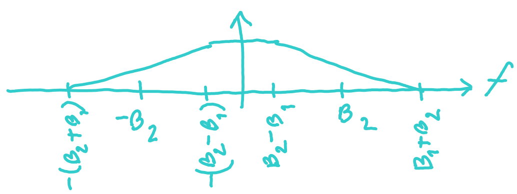


Of course, $G_1(f)$ and $G_2(f)$ in this question won't look like these. We draw them this way to make the conclusion easier to see.

Because we have a multiplication in the time domain,
we have a convolution in the frequency domain.
The convolution with an impulse is easy:
$$G_1(f) * \delta(f - f_0) = G_1(f - f_0).$$
So, we simply have replicas of $G_1(f)$ at all the impulses' locations of $G_2(f)$. Hence, the highest freq. component is at $B_1 + B_2$ and
the lowest freq. component is at $-B_1 - B_2$.
Alternatively, one can look at the convolution of two rectangular functions:



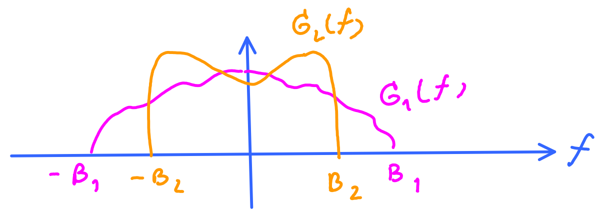
We saw similar convolution as a video earlier in the semester. The result is



Hence, for $\text{sinc}^2(100\pi t)$, $B = 50+50 = 100$ Hz

(c) Observe that for signals $g_1(t)$ bandlimited to B_1 and $g_2(t)$ " " B_2,

their linear combination $c_1 g_1(t) + c_2 g_2(t)$ is bandlimited to $\max\{B_1, B_2\}$.



So, for $\text{sinc}(100\pi t) + \text{sinc}(50\pi t)$, $B = \max\{50, 25\} = 50$ Hz

(d) Use the observation from parts (b) and (c).

For $\text{sinc}(100\pi t) + 3\,\text{sinc}^2(60\pi t)$, $B = \max\{50, 2\times 60\} = 120$ Hz.

(e) Use the same observation as in part (b).

For $\text{sinc}(50\pi t)\,\text{sinc}(100\pi t)$, $B = 25 + 50 = 75$ Hz.

Now that we know the max freq. B of our signals:

The Nyquist sampling rate is $2\times B$.

The Nyquist sampling interval is $\frac{1}{2B}$.

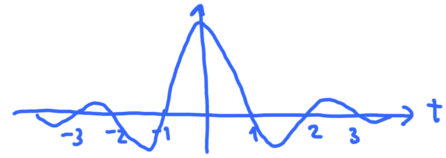
The table below summarizes the answers for this question:

	f_{max}	$R_{Nyquist}$ [Sa/s]	$T_{Nyquist}$ [Sec]
(a)	50	100	0.01
(b)	100	200	0.005
(c)	50	100	0.01
(d)	60	120	1/120
(e)	75	150	1/150

Q4 Sinc Reconstruction of Sinc

Thursday, August 30, 2012 1:51 PM

The signal under consideration is $g(t) = \text{sinc}(\pi t)$.



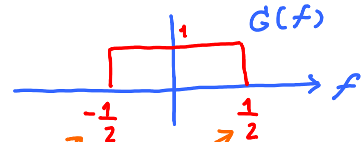
note that, in MATLAB, this function is implemented by sinc(t) because the built-in MATLAB sinc function has already included the π.

(a) The Fourier transform

$g(0) = 1$. So, $\int_{-\infty}^{\infty} G(f)\, df = 1$.

To have area = 1, the hight of the rectangular pulse must be 1 as well.

$g(t) = \text{sinc}(\pi t) = \text{sinc}\left(2\pi \frac{1}{2} t\right) \xrightarrow{\mathcal{F}}$

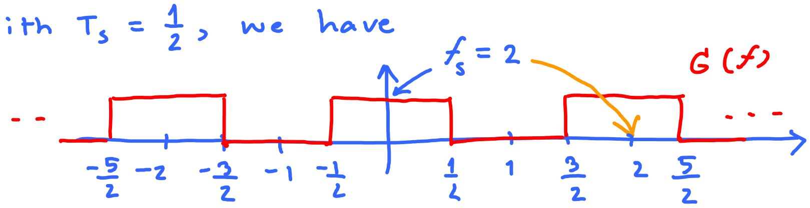


(b) The Nyquist sampling rate is given by $2 \times f_{max} = 2 \times \frac{1}{2} = 1$ samble/sec.

(c) In class, we have seen that

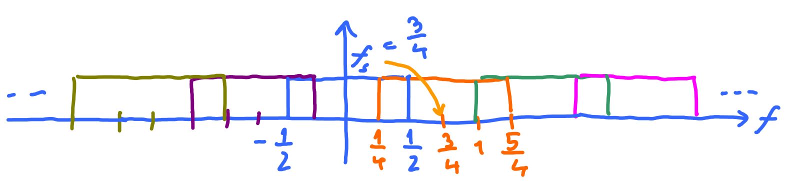
$$G_s(f) = f_s \sum_{n=-\infty}^{\infty} G(f - n f_s) \quad \text{where } f_s = \frac{1}{T_s}.$$

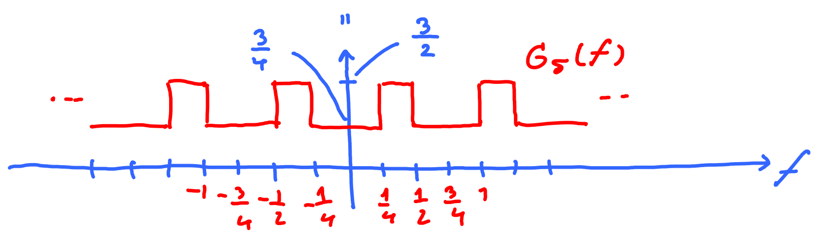
(c.i) With $T_s = \frac{1}{2}$, we have



Periodic with "period" = 2.

(c.ii) With $T_s = \frac{4}{3}$, we have





Periodic with "period" = $\frac{3}{4}$.

(d.i)

With $T_s = 1$, $g[n] = g(nT_s) = g(n\times 1) = g(n) = \text{sinc}(\pi n)$.

(d.i.i) From the plot of $\text{sinc}(\pi n)$ drawn earlier, we have

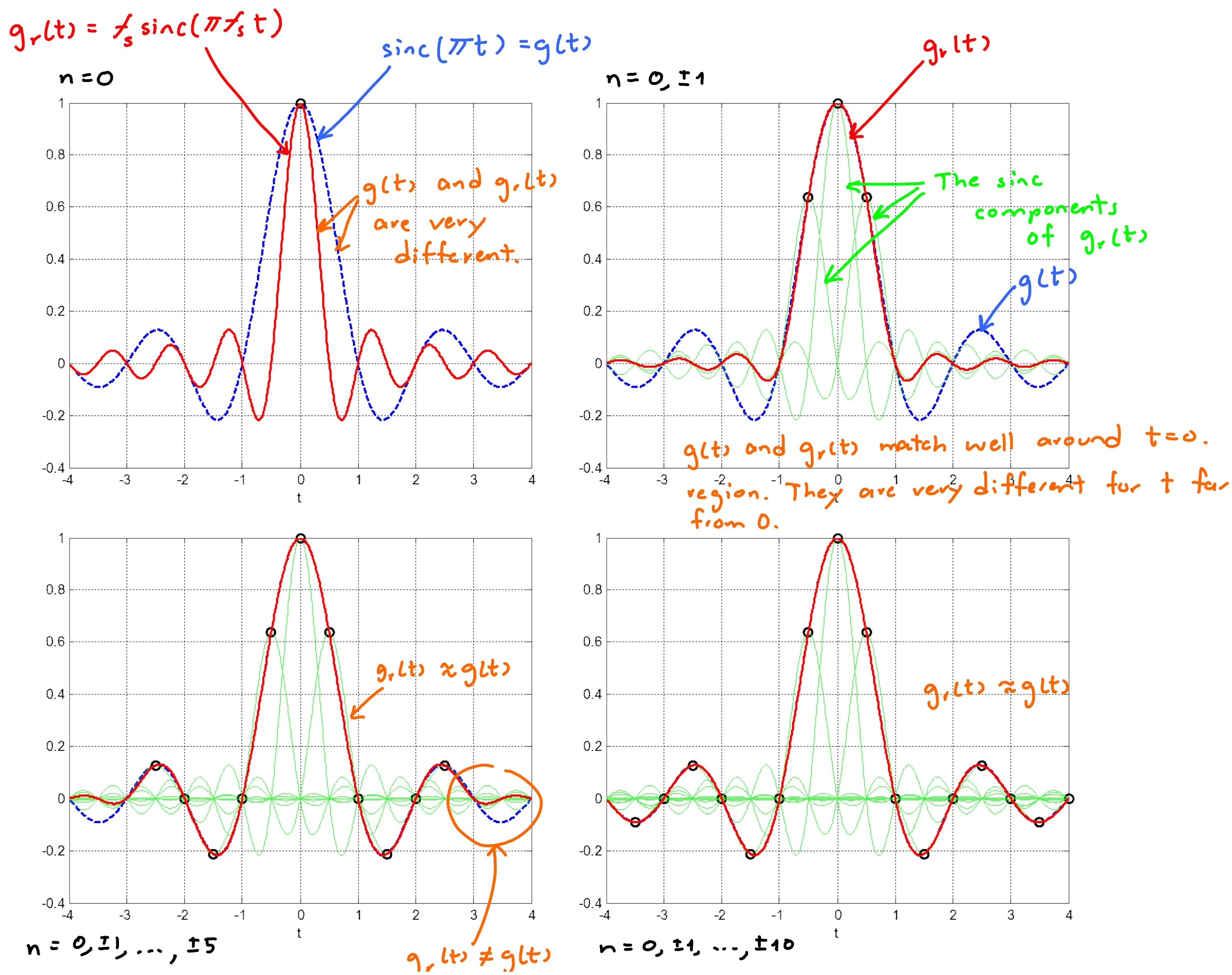
$$g[n] = \begin{cases} 1, & n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

(d.i.ii) Because $g[n] = 0$ when $n \neq 0$,

$$g_r(t) = g[0]\,\text{sinc}(\pi t) = 1 \times \text{sinc}(\pi t) = \text{sinc}(\pi t) = g(t)$$

Note that with $T_s = 1$, we have $f_s = 1$ which is the same as the Nyquist sampling rate. Therefore, we are at the borderline of the successful reconstruction.

(d.ii) Reminder: MATLAB's sinc function is $\text{sinc}(x) = \sin(\pi x)/\pi x$, which is different from $\text{sinc}(x) = \sin(x)/x$ that we defined for our class. Therefore, when we use MATLAB to plot $\text{sinc}(\pi t)$, we do not put the "π" in the formula. MATLAB will automatically insert the "π" for us.



$n = 0, \pm 1, \ldots, \pm 5$ $\quad$ $g_r(t) \neq g(t)$ $\quad$ $n = 0, \pm 1, \ldots, \pm 10$

Observation: As we increase the number of terms in the summation, $g(t)$ is better approximated by $g_r(t)$.